The Mathematics Of Voting

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This resource is designed to be used with the Konstantinov system, and provides an introduction into the main results in the mathematical study of elections. All necessary definitions are introduced as they are required, and questions should be completed in order.

This sheet is a work in progress in that more sections will be added over time. The current two sections are internally complete though.

1. Social Choice Theory and May's Theorem

Definition 1 (Social Choice Function) *Let S be a finite set of options between which voters will choose. Elements of this set are called alternatives, and we denote the number of alternatives, that is $|S|$, by m.*

A social choice function is a function $f : \mathrm{Bij}(\{1, ..., m\}, S)^n \to \mathcal{P}(S) \setminus \varnothing$ where n is the number of voters.

A particular element of $\mathrm{Bij}(\{1, ..., m\}, S)^n$ is called a profile while a particular element of $\mathcal{P}(S) \setminus \varnothing$ is called a social choice.

Remark 1 *The above definition is designed to introduce the necessary formalisms to bring our real world understanding of elections into the world of pure mathematics.*

The alternatives are the elements which are being chosen between. This may be candidates of an election, or which movie to watch in a group of friends.

The set $\mathrm{Bij}(\{1, ..., m\}, S)$ is the set of bijections from $\{1, ..., m\}$ to S. As such, each element of this set represents a ranked ballot paper. That is, for each of the possible ranks from 1 down to the number of alternatives is assigned a particular alternative. A social choice function has domain $\mathrm{Bij}(\{1, ..., m\}, S)^n$ as this is a collection of n votes, one for each voter. Thus the profile can be thought of as the collection of all votes in an election.

The output of the social choice function can be thought of as a collection of winners of the election, with the potential possibility of ties. If we wish to prohibit ties, we will sometimes instead take the codomain to be S instead.

Remark 2 *When only choosing between two alternatives we typically use $S = \{-1, 1\}$ and identify each bijection in $\mathrm{Bij}(\{1, 2\}, S)$ by simply the image of the first preference 1. We also use the bijection $\mathcal{P}(S) \setminus \varnothing$ given by $\{-1, 1\} \mapsto 0$, $\{-1\} \mapsto -1$, $\{1\} \mapsto 1$ such that 0 represents a tie.*

As such, our social choice function becomes a function $f : \{-1, 1\}^n \to \{-1, 0, 1\}$ where -1 and 1 are the alternatives and the output is the winning alternative except in the case of 0, which represents a tie.

If we wish to exclude the possibility of tie we use a function $f : \{-1, 1\}^n \to \{-1, 1\}$.

Definition 2 *Define the function $\mu : \{-1, 1\}^n \to \mathbb{Z}_{\geq 0}$ such that for a given vector $\vec{x} \in \{-1, 1\}^n$ we let $\mu(\vec{x})$ denote the number of components of the vector which are 1. This is such that for a given voter profile $\vec{x}$, $\mu(\vec{x})$ is the number of voters for the alternative denoted by 1.*

Question 1

a) Explain why for any $\vec{x} \in \{-1,1\}^n$ we have $\mu(-\vec{x}) = n - \mu(\vec{x})$. What does $\mu(-\vec{x})$ represent in the context of a voter profile?

b) Show that

$$\mu(\vec{x}) = \frac{1}{2}\sum_{i=1}^{n} x_i + \frac{n}{2}.$$

c) Show that $\mu(\vec{x}) > \frac{n}{2} \iff \mu(-\vec{x}) < \frac{n}{2}$.

d) Prove that $\mu(\vec{x}) > \mu(-\vec{x}) \iff \mu(\vec{x}) > \frac{n}{2}$ and similarly that $\mu(\vec{x}) < \mu(-\vec{x}) \iff \mu(\vec{x}) < \frac{n}{2}$.

Question 2 In terms of m and n, how many possible voter profiles are there in a ranked choice election?

Question 3 How many possible functions $\{-1,1\}^n \to \{-1,1\}$ and $\{-1,1\}^n \to \{-1,0,1\}$ are there?

Definition 3 *A social choice function is called anonymous if $f(\vec{x}) = f(\vec{y})$ whenever $\vec{x}$ is a permutation of $\vec{y}$.*

Definition 4 *A social choice function is called neutral if $-f(\vec{x}) = f(-\vec{x})$ for any $\vec{x}$.*

Definition 5 *A social choice function is called monotone if $f(\vec{x}) \geq f(\vec{y})$ for any $\vec{x} = (x_1, x_2, ..., x_n)$ and $\vec{y} = (y_1, y_2, ..., y_n)$ satisfying $x_k \geq y_k$ for all $k \in \{1, ..., n\}$.*

Discussion 1 *Reason through and explain what each of the above properties (anonymous, neutral, monotone) mean for our social choice function in terms of expressing voter preference and hence why they may be desirable.*

Question 4 How many possible anonymous social choice functions $\{-1,1\}^n \to \{-1,1\}$ are there?

Question 5 Suppose $f : \{-1,1\}^n \to \{-1,0,1\}$ is a social choice function between two alternatives which prohibits ties. Prove that if f is both anonymous and monotone, then

$$\mu(\vec{x}) \geq \mu(\vec{y}) \implies f(\vec{x}) \geq f(\vec{y}).$$

That is, more votes in favour for an alternative cannot result in a less favourable result for that alternative.

Definition 6 *A social choice function $f : \{-1,1\}^n \to \{-1,0,1\}$ is called a majority vote function if*

$$f(\vec{x}) = 1 \iff \mu(\vec{x}) > \frac{n}{2} \quad \text{and} \quad f(\vec{x}) = 0 \iff \mu(\vec{x}) = \frac{n}{2}.$$

That is, an alternative is elected if and only if they get more than half the total number of votes, and a tie only occurs when both candidates get the same number of votes.

Question 6 (May's Theorem) Suppose $f : \{-1,1\}^n \to \{-1,0,1\}$ is a social choice function between two alternatives and that n is an odd number. Prove that if f is anonymous, monotone, and neutral, then f must be the majority vote.

2. More Alternatives and Condorcet's Paradox

Definition 7 (Plurality Voting) *Plurality voting extends the majority vote in the case of two alternatives by declaring the winner as the candidate with the greatest number of first preference votes.*

Definition 8 (Borda Count) *A Borda count, for each voter, assigns points to each alternative depending on their ranking. In particular, $m-1$ poinst are given to the candidate of the first preference, $m-2$ for the second preference, and so on until the lowest ranked candidate who receives 0 points. The winning alternative is then the alternative with the greatest total number of points. If multiple alternatives are tied for the same number of points, the election itself is a tie.*

Definition 9 (Positional Voting System) *A positional voting system generalises the Borda count by allowing an arbitrary sequence of point values to be used.*

Remark 3 *Both plurality voting and the Borda count are special cases of a positional voting system.*

Question 7 For the profile

1.	A	B	C	A
2.	B	C	D	D
3.	C	D	B	B
4.	D	A	A	C

determine the winner by Borda count and by positional voting with points assigned as $0, 1, 2, 5$.

Question 8 Construct a voter profile which yields a different winner when counted with plurality voting and the Borda count? How "small" can such a profile be with respect to the number of alternatives and voters?

Definition 10 *Given a profile of preferential votes, we can restrict to two alternatives by simply ignoring all preferences for other alternatives. If alternative A beats alternative B in a majority vote, we then write $A \succ B$ or $B \prec A$.*

Definition 11 *If there exists an alternative A such that $A \succ B$ for all alternatives $B \in S$ then A is called the Condorcet winner.*

Question 9 (Condorcet's Paradox) Construct a profile of three voters and three alternatives A, B, C in which $A \prec B$, $B \prec C$ and $C \prec A$, thus proving that $\prec$ is not a partial order and that a Condorcet winner does not necessarily exist.

Question 10 Show that the counterexample from Question 9 is minimal in the sense that any profile with fewer than three alternatives or fewer than three voters must have a Condorcet winner.

Question 11 Show that for three voters and three alternatives, the probability of a Condocet winner existing is $\frac{17}{18}$.

Hint: Try and use what you learned when working through Question 9, in particular, try to figure out what properties a profile without a Condorcet winner must have, and then count the number of profiles with such properties.